Neighborhood semantics and proof theory for infinitary intuitionistic logic

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Outline

Background

- The most known semantics for intuitionistic logic is the informal interpretation of connectives, called *BHK* which is based on the conception of truth as a constructive mathematical proof.
- Many formal semantics were developed: Kripke relational semantics, topological semantics, Beth semantics, algebraic semantics... Intuitionistic propositional logic is sound and complete for all these semantics.
- Infinitary intuitionistic logic is intuitionistic logic extended with countable disjunctions and conjunctions. The axiom scheme:

$$\bigwedge_{k>0}(A_k \vee B) \to \bigwedge_{k>0} A_k \vee B$$

is not valid in the algebraic semantics, yet it holds in Kripke models extended with a natural truth condition for infinitary conjunction:

$$x \vDash \bigwedge_{k>0} A_k \text{ iff } x \vDash A_k \text{ for every } k > 0$$

- The problem depends on the fact that Kripke frames for intuitionistic logic are partial orders, which correspond to Alexandroff topologies, i.e. topologies closed under arbitrary intersections. Kripke semantics is not adequate for infinitary intuitionistic logic.

The sequent calculus **G3i**$_\omega$

$$\frac{}{p, \Gamma \Rightarrow \Delta, p}\ Ax \qquad \frac{}{\bot, \Gamma \Rightarrow \Delta}\ L\bot$$

$$\frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\ L\wedge \qquad \frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}\ R\wedge$$

$$\frac{A, \Gamma \Rightarrow \Delta \qquad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta}\ L\vee \qquad \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B}\ R\vee$$

$$\frac{A \rightarrow B, \Gamma \Rightarrow \Delta, A \qquad B, \Gamma \Rightarrow \Delta}{A \rightarrow B, \Gamma \Rightarrow \Delta}\ L\rightarrow \qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow \Delta, A \rightarrow B}\ R\rightarrow$$

$$\frac{\bigwedge_{k>0} A_k, A_k, \Gamma \Rightarrow \Delta}{\bigwedge_{k>0} A_k, \Gamma \Rightarrow \Delta}\ L\bigwedge_k \qquad \frac{\{\Gamma \Rightarrow A_k \mid k > 0\}}{\Gamma \Rightarrow \Delta, \bigwedge_{k>0} A_k}\ R\bigwedge$$

$$\frac{\{A_k, \Gamma \Rightarrow \Delta \mid k > 0\}}{\bigvee_{k>0} A_k, \Gamma \Rightarrow \Delta}\ L\bigvee \qquad \frac{\Gamma \Rightarrow \Delta, \bigvee_{k>0} A_k, A_k}{\Gamma \Rightarrow \Delta, \bigvee_{k>0} A_k}\ R\bigvee_k$$

Every rule except for $R\bigwedge$ and $R\rightarrow$ is invertible. Weakening, contraction and cut are admissible Negri (2020).

Outline

The presentation is divided in three main sections:

1. Starting from algebraic semantics, we establish completeness with respect to topological semantics for countable fragments of the language. We finally introduce a neighborhood semantics for intuitionistic infinitary logic:

   $$\textit{Algebraic} \rightarrow \textit{Topological} \rightarrow \textit{Neighborhood}$$

2. We extract a labelled sequent calculus from neighborhood semantics and we show its structural properties, namely admissibility of weakening, contraction, cut and *invertibility of every rule*.
3. We present a neighborhood semantics for infinitary S4 modal logic and we exploit it in order to obtain a labelled sequent calculus. We then prove an extension of the usual Gödel-McKinsey-Tarski embedding to the infinitary setting.

Algebraic semantics

A complete Heyting algebra is a structure $\langle H, \wedge, \vee, \rightarrow, 0, 1\rangle$, where

- $\langle H, \wedge, \vee, 0, 1\rangle$ is a bounded distributive complete lattice
- For every $x, y \in H$ there is $x \rightarrow y$ s. t. for every $z \in H$:

$$z \wedge x \leqslant y \text{ iff } z \leqslant x \rightarrow y$$

- The following distributivity law holds for every $x \in H$ and for every $B \subseteq H$:

$$x \wedge \bigvee_{b \in B} b = \bigvee_{b \in B} (x \wedge b)$$

An algebraic model for infinitary intuitionistic logic is a pair $\langle \mathcal{H}, v\rangle$, where:

- $\mathcal{H}$ is a complete Heyting algebra.
- $v : AT \rightarrow H$ is a valuation function.

A formula A is valid in the algebraic semantics for infinitary intuitionistic logic iff $v(A) = 1$ for every algebraic model.

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The notion of environment

The language of infinitary intuitionistic logic built from atomic formulas, finite connectives and infinitary (countable) conjunctions and disjunctions **is not countable** (consider the fact the countable subsets of a countable set are more than countable and to each of them can be assigned a different infinitary conjunction).

For every subset of formulas Γ, the **environment** $\mathcal{E}(\Gamma)$ (see Minari (2016)) is the minimum subset of formulas such that:

- $AT \subseteq \mathcal{E}(\Gamma), \bot \in \mathcal{E}(\Gamma)$ and $\mathcal{E}(\Gamma)$ contains every subformula of Γ
- $\mathcal{E}(\Gamma)$ is closed under finite conjunctions, disjunctions and implications
- The following closure condition holds: if $A, \bigvee_{k>0} B_k \in \mathcal{E}(\Gamma)$, then $\bigvee_{k>0}(A \wedge B_k) \in \mathcal{E}(\Gamma)$.

Lemma. For every $A \in FOR$, $\mathcal{E}(A)$ is countable. For every countable subset of formulas Γ, the environment $\mathcal{E}(\Gamma)$ is countable.

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Topological semantics

A topological space is a pair $\langle W, \tau \rangle$ where W is a set and τ is a family of subsets of W which contain $\emptyset$ and W, is closed under arbitrary unions and under finite intersections. The elements of τ are referred to as open sets. For every $x \in W$, we denote with τ_x the open sets which contain x.

A topological model for a countable fragment Σ of infinitary intuitionistic logic is a triple $\langle W, \tau, v \rangle$, where:

- $\langle W, \tau \rangle$ is a topological space.
- $v : AT \to \tau$ is the valuation function and for every formula in $\mathcal{E}(\Sigma)$:
  - $v(A \wedge B) = v(A) \cap v(B)$
  - $v(A \vee B) = v(A) \cup v(B)$
  - $v(A \to B) = \bigcup \{a \in \tau \mid a \subseteq v(A)^c \cup v(B)\} = Int(v(A)^c \cup v(B))$
  - $v(\bigvee_{k>0} A_k) = \bigcup_{k>0} v(A_k)$
  - $v(\bigwedge_{k>0} A_k) = \bigcup \{a \in \tau \mid a \subseteq \bigcap_{k>0} v(A_k)\} = Int(\bigcap\limits_{k>0} v(A_k))$

Notice that in the case of the valuation of $\to$ and $\bigwedge$ we can restrict to the open sets of a base of the topology, see Valentini (1994). A formula A is valid in topological semantics for intuitionistic infinitary logic iff for every topological model $v(A) = W$.

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From algebraic semantics to topological semantics

- Completeness for countable fragments of intuitionistic infinitary logic with respect to topological semantics is referred to only in a paper by Nadel (1978).
- The key lemma to establish completeness for intuitionistic infinitary logic is a reformulation of a result obtained by Valentini (1994) in order to prove completeness with respect to first order intuitionistic logic.

Definition. For every Heyting algebra $\mathcal{H}$ and every subset B of $\mathcal{H}$ with a supremum, a filter F respects B if, whenever $\sup B \in F$, there is $b \in B \cap F$.

Theorem. Let $\mathcal{H}$ be a Heyting algebra and x, y two elements in $\mathcal{H}$ such that $x \not\leq y$. Let $B_1, ..., B_n, ...$ a countable quantity of subsets of $\mathcal{H}$ which have a supremum, then there is a prime filter F which respects every subset B_i and $x \in F$, $y \notin F$.

Proof. Valentini (1994).

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Proof. Valentini (1994).

Theorem. (Topological completeness) For every A, if $\models_{Top} A$, then $G3i_\omega \vdash \Rightarrow A$.

Proof. (Sketch). Let us assume that $G3i_\omega \nvdash \Rightarrow A$. We consider the Lindenbaum algebra $\mathcal{A}$ obtained by $\mathcal{E}(A)$. We fix the valuation $v(p) = [p]$ for every $p \in AT$. We immediately obtain that $v(A) \neq [\bot \to \bot]$.

$|\mathcal{E}(A)_{/\sim}|$ is countable, thus we enumerate the subsets $B_1, B_2, ..., B_n, ...$ of $\mathcal{E}(A)_{/\sim}$ such that $\bigvee B_1, ..., \bigvee B_n, ...$ are in $\mathcal{A}$.
Let $Pt(\mathcal{A}) = \{P \text{ prime filter} \mid P \text{ respects } B_i\}$. We consider the function $ext : \mathcal{A} \to Pt(\mathcal{A})$ s.t. $ext([C]) = \{P \in Pt(\mathcal{A}) \mid [C] \in P\}$. We consider the set:

$$\mathcal{B}_{\tau_{\mathcal{A}}} = \{ext([C]) \mid C \in \mathcal{E}(A)\}$$

as a base for the topology $\langle Pt(\mathcal{A}), \tau_{\mathcal{A}} \rangle$. The next result makes essential use of the Theorem for Heyting algebras with countable suprema.

Theorem. $ext : \mathcal{A} \to Pt(\mathcal{A})$ is an injective morphism which preserves sups and infs.

Taking as a valuation $v'(p) = ext([p])$ we have obtained a topological countermodel for A.

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Neighborhood semantics

A neighborhood frame for intuitionistic infinitary logic is a pair $\langle W, N \rangle$ where W is a non empty set and $N : W \to \mathcal{P}(\mathcal{P}(W))$ s. t.:

- If $a \in N(x)$ and $b \in N(x)$, then $a \cap b \in N(x)$
- If $a \in N(x)$ and $a \subseteq b$, then $b \in N(x)$
- $W \in N(x)$ and $\emptyset \notin N(x)$
- For every $x \in W$, if $a \in N(x)$, then $x \in a$
- If $a \in N(x)$, then $m(a) = \{y \mid a \in N(y)\} \in N(x)$

Definition. A neighborhood model for intuitionistic infinitary logic is a triple $\langle W, N, v \rangle$ where $v : AT \to \mathcal{P}(W)$ s.t. for every $p \in AT$, if $x \in v(p)$, then $v(p) \in N(x)$.

We introduce some abbreviations:

- Let $a \Vdash^{\forall} B$, *a forces universally B*, indicate that every element in a satisfies B.
- $x \vDash A \supset B$ indicates *if* $x \vDash A$*, then* $x \vDash B$.
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Neighborhood semantics (cont.)

The truth conditions for neighborhood semantics are inductively defined:

- $x \vDash p$ iff $x \in v(p)$
- $x \vDash A \wedge B$ iff $x \vDash A$ and $x \vDash B$
- $x \vDash A \vee B$ iff $x \vDash A$ or $x \vDash B$
- $x \vDash A \rightarrow B$ iff $\exists a \in N(x)(a \Vdash^{\forall} A \supset B)$
- $x \vDash \bigvee_{k>0} A_k$ iff $x \vDash A_k$ for some $k > 0$
- $x \vDash \bigwedge_{k>0} A_k$ iff $\exists a \in N(x)(a \Vdash^{\forall} \&_{k>0} A_k)$

Lemma. For every $A \in FOR$, if $x \vDash A$, then there is $b \in N(x)$ s.t. $b \Vdash^{\forall} A$.

Proof. By transfinite induction on the degree of A, exploiting the properties of infinitary intuitionistic neighborhood frames.

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From topological to neighborhood semantics

Given a topological space $\langle W, \tau \rangle$ its *associated neighborhood system* is $\langle W, N_\tau \rangle$, where $N_\tau(x) = \{a \in \mathcal{P}(W) \mid \exists b \in \tau_x (b \subseteq a)\}$.

Theorem. The neighborhood system associated $\langle W, N_\tau \rangle$ is an intuitionistic infinitary frame.

Theorem. Let $\mathcal{M} = \langle W, \tau, v \rangle$ be a topological model. Let $\langle W, N_\tau \rangle$ be its associated neighborhood frame. We consider the neighborhood model $\mathcal{N} = \langle W, N_\tau, v_\tau \rangle$, with $v_\tau(p) = v(p)$, then:

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Theorem. (Neigh. completeness) $G3i_\omega \vdash \Rightarrow A$ iff $\vDash_{\mathcal{N}} A$.
Proof. From left to right we argue by transfinite induction on the height of derivation distinguishing cases according to the last rule applied. From right to left we prove the contrapositive. If $G3i_\omega \nvdash \Rightarrow A$, then by topological completeness there is a topological model $\langle W, N, v \rangle$ such that $v(A) \neq W$, thus its associated neighborhood model is a countermodel to A.

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From neighborhood semantics to rules

We introduce a propositional base $G3C_\omega$ Negri (2020).

$$\frac{}{x : p, \Gamma \Rightarrow \Delta, x : p}\ {\scriptstyle Ax} \qquad \frac{}{x : \bot, \Gamma \Rightarrow \Delta}\ {\scriptstyle L\bot}$$

$$\frac{x : A, x : B, \Gamma \Rightarrow \Delta}{x : A \wedge B, \Gamma \Rightarrow \Delta}\ {\scriptstyle L\wedge} \qquad \frac{\Gamma \Rightarrow \Delta, x : A \qquad \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : A \wedge B}\ {\scriptstyle R\wedge}$$

$$\frac{x : A, \Gamma \Rightarrow \Delta \qquad x : B, \Gamma \Rightarrow \Delta}{x : A \vee B, \Gamma \Rightarrow \Delta}\ {\scriptstyle L\vee} \qquad \frac{\Gamma \Rightarrow \Delta, x : A, x : B}{\Gamma \Rightarrow \Delta, x : A \vee B}\ {\scriptstyle R\vee}$$

$$\frac{\Gamma \Rightarrow \Delta, x : A \qquad x : B, \Gamma \Rightarrow \Delta}{x : A \supset B, \Gamma \Rightarrow \Delta}\ {\scriptstyle L\supset} \qquad \frac{x : A, \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : A \supset B}\ {\scriptstyle R\supset}$$

$$\frac{x : \underset{k>0}{\&} A_k, x : A_k, \Gamma \Rightarrow \Delta}{x : \underset{k>0}{\&} A_k, \Gamma \Rightarrow \Delta}\ {\scriptstyle L\&} \qquad \frac{\{\Gamma \Rightarrow \Delta, x : A_k \mid k > 0\}}{\Gamma \Rightarrow \Delta, x : \underset{k>0}{\&} A_k}\ {\scriptstyle R\&}$$

$$\frac{\{x : A_k, \Gamma \Rightarrow \Delta \mid k > 0\}}{x : \bigvee_{k>0} A_k, \Gamma \Rightarrow \Delta}\ {\scriptstyle L\bigvee} \qquad \frac{\Gamma \Rightarrow \Delta, x : \bigvee_{k>0} A_k, x : A_k}{\Gamma \Rightarrow \Delta, x : \bigvee_{k>0} A_k}\ {\scriptstyle R\bigvee}$$

From neighborhood semantics to rules (cont.)

We add rules for $\Vdash^{\forall}$:

$$\frac{x \in a, a \Vdash^{\forall} A, x : A, \Gamma \Rightarrow \Delta}{x \in a, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta}\ L\Vdash^{\forall} \qquad \frac{y \in a, \Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, a \Vdash^{\forall} A}\ R\Vdash^{\forall}, y \textit{ fresh}$$

Rules for $\rightarrow$ and $\bigwedge$ are obtained in two steps through those of $\supset$, & and $\Vdash^{\forall}$:

$$\frac{a \in N(x), a \Vdash^{\forall} \underset{k>0}{\&} A_k, \Gamma \Rightarrow \Delta}{x : \bigwedge_{k>0} A_k, \Gamma \Rightarrow \Delta}\ L\bigwedge, a \textit{ fresh} \qquad \frac{a \in N(x), \Gamma \Rightarrow \Delta, x : \bigwedge_{n>0} A_n, a \Vdash^{\forall} \underset{k>0}{\&} A_k}{a \in N(x), \Gamma \Rightarrow \Delta, x : \bigwedge_{k>0} A_n}\ R\bigwedge$$

$$\frac{a \in N(x), a \Vdash^{\forall} A \supset B, \Gamma \Rightarrow \Delta}{x : A \rightarrow B, \Gamma \Rightarrow \Delta}\ L\rightarrow, a \textit{ fresh} \qquad \frac{a \in N(x), \Gamma \Rightarrow \Delta, x : A \rightarrow B, a \Vdash^{\forall} A \supset B}{a \in N(x), \Gamma \Rightarrow \Delta, x : A \rightarrow B}\ R\rightarrow$$

We now add the auxiliary rules which are obtained by converting the frame conditions into rules as usual. Due to the property of closure under supersets some conditions can be streamlined.
In particular $W \in N(x)$ is equivalent to $N(x) \neq \emptyset$ and the condition imposed on the valuation function can be reformulated: if $x \in v(p)$, then there is $a \in N(x)$ s.t. $a \Vdash^{\forall} p$.

From neighborhood semantics to rules (cont.)

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$$\frac{a \in N(x), a \Vdash^{\forall} \underset{k>0}{\&} A_k, \Gamma \Rightarrow \Delta}{x : \bigwedge\limits_{k>0} A_k, \Gamma \Rightarrow \Delta}\ L\bigwedge, a \textit{ fresh} \qquad \frac{a \in N(x), \Gamma \Rightarrow \Delta, x : \bigwedge_{n>0} A_n, a \Vdash^{\forall} \underset{k>0}{\&} A_k}{a \in N(x), \Gamma \Rightarrow \Delta, x : \bigwedge\limits_{k>0} A_n}\ R\bigwedge$$

$$\frac{a \in N(x), a \Vdash^{\forall} A \supset B, \Gamma \Rightarrow \Delta}{x : A \rightarrow B, \Gamma \Rightarrow \Delta}\ L\rightarrow, a \textit{ fresh} \qquad \frac{a \in N(x), \Gamma \Rightarrow \Delta, x : A \rightarrow B, a \Vdash^{\forall} A \supset B}{a \in N(x), \Gamma \Rightarrow \Delta, x : A \rightarrow B}\ R\rightarrow$$

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Auxiliary rules

$$\frac{x : p, a \in N(x), a \Vdash^{\forall} p, \Gamma \Rightarrow \Delta}{x : p, \Gamma \Rightarrow \Delta}\ \textit{Mon}, a \text{ fresh}$$

$$\frac{a \in N(x), x \in a, \Gamma \Rightarrow \Delta}{a \in N(x), \Gamma \Rightarrow \Delta}\ \textit{Ref} \qquad \frac{a \in N(x), \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \textit{Nondeg}, a \text{ fresh}$$

$$\frac{a \in N(x), b \in N(x), a \cap b \in N(x), \Gamma \Rightarrow \Delta}{a \in N(x), b \in N(x), \Gamma \Rightarrow \Delta}\ C \qquad \frac{x \in a \cap b, x \in a, x \in b, \Gamma \Rightarrow \Delta}{x \in a \cap b, \Gamma \Rightarrow \Delta}\ L\cap$$

$$\frac{a \in N(x), m(a) \in N(x), \Gamma \Rightarrow \Delta}{a \in N(x), \Gamma \Rightarrow \Delta}\ \textit{Trs} \qquad \frac{x \in m(a), a \in N(x), \Gamma \Rightarrow \Delta}{x \in m(a), \Gamma \Rightarrow \Delta}\ \textit{Lm}$$

$G3I_\omega = G3C_\omega +$ rules for $\rightarrow$, $\bigwedge$ and $\Vdash^{\forall}$ $+$ Auxiliary rules.

Derivations in infinitary sequent calculi

Formal definition of the notion of *derivation* $\mathcal{D}$ and *height* $ht(\mathcal{D})$ and its *end-sequent*:

1. Any sequent $\Gamma \Rightarrow \Delta$, where some atomic formula $x : p$ occurs in both Γ and Δ, is a derivation, of *height* 0 and with *end-sequent* $\Gamma \Rightarrow \Delta$.
2. Any sequent $\Gamma \Rightarrow \Delta$ where $\perp$ occurs in Γ and Δ, is a derivation, of *height* 1 and with end-sequent $\Gamma \Rightarrow \Delta$.
3. If each $\mathcal{D}_n$ is a derivation, of height α_n, with end-sequent $\Gamma_n \Rightarrow \Delta_n$ and

   $$\frac{\ldots \quad \Gamma_n \Rightarrow \Delta_n \quad \ldots}{\Gamma \Rightarrow \Delta}\ R$$

   is an inference (i.e. an instance of a rule), then

   $$\frac{\ldots \quad \dfrac{\mathcal{D}_n}{\Gamma_n \Rightarrow \Delta_n} \quad \ldots}{\Gamma \Rightarrow \Delta}\ R$$

   is a derivation, of *height* the countable ordinal $sup_n(\alpha_n) + 1$ and with *end-sequent* $\Gamma \Rightarrow \Delta$.

- Each derivation has a countable ordinal *height* (the successor of the supremum of the heights of its immediate subderivations).
- If $\mathcal{D}'$ is a subderivation of $\mathcal{D}$, then $ht(\mathcal{D}') < ht(\mathcal{D})$.

Degree of labelled formulas

The *label* of a formula $x : A$ is x, the *label* of a formula $a \Vdash^{\forall} A$ is a. The label of a formula ϕ is denoted by $l(\phi)$. The *pure part* of a labelled formula ϕ is obtained removing from ϕ the *label* and the forcing relation and is denoted by $p(\phi)$. The notion of *weight* is defined for *labels* and *pure parts* of formulas:

- For every x and for every a, $w(x) = 0$ and $w(a) = 1 + n(\cap) + n(m)$, where $n(\cap)$ is the number of signs of intersections in a and $n(m)$ is the number of the occurrences of m in a.
- The *weight* of a pure formula A, $w(A)$ is defined as follows:
  - $w(p) = w(\bot) = 1$
  - $w(A \circ B) = sup(\{w(A), w(B)\}) + 1$, where $\circ \in \{\wedge, \vee, \supset\}$
  - $w(A \to B) = sup(\{w(A), w(B)\}) + 2$
  - $w(\bigvee_{k>0} A_k) = sup_k(w(A_k)) + 1$
  - $w(\underset{k>0}{\&} A_k) = sup_k(w(A_k)) + 1$
  - $w(\bigwedge_{k>0} A_k) = sup_k(w(A_k)) + 2$

The *degree* of a labelled formula ϕ is an ordered pair $deg(\phi) = (w(p(\phi)), w(l(\phi)))$. For relational formulas we stipulate $deg(x \in a) = deg(a \in N(x)) = (0, w(a))$. *Degrees* of labelled formulas are ordered lexicographically.

Structural properties of $G3I_\omega$

The structural properties are established by transfinite induction on the height of derivation. The formula weight is defined to reflect the meaning explanation.

Theorem. Every rule is invertible and the rules of weakening and contraction are height-preserving admissible.
Proof. Routine by induction on the height of derivations.

Lemma. The following rules are admissible in $G3I_\omega$.

$$\frac{a \Vdash^{\forall} A \vee B, \Gamma \Rightarrow \Delta}{a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta}\,Mon_{\vee} \qquad \frac{a \cap b \Vdash^{\forall} A \wedge B, \Gamma \Rightarrow \Delta, y : A}{a \Vdash^{\forall} A, b \Vdash^{\forall} B, \Gamma \Rightarrow \Delta}\,Mon_{\wedge} \qquad \frac{m(a) \Vdash^{\forall} A \to B, \Gamma \Rightarrow \Delta}{a \Vdash^{\forall} A \supset B, \Gamma \Rightarrow \Delta}\,Mon_{\to}$$

$$\frac{a \Vdash^{\forall} \bigvee_{k>0} A_k, \Gamma \Rightarrow \Delta}{a \Vdash^{\forall} A_k, \Gamma \Rightarrow \Delta}\,Mon_{\bigvee} \qquad \frac{m(a) \Vdash^{\forall} \bigwedge_{k>0} A_k, \Gamma \Rightarrow \Delta}{a \Vdash^{\forall} \mathop{\&}_{k>0} A_k, \Gamma \Rightarrow \Delta}\,Mon_{\bigwedge}$$

Proof. By induction on the height of derivations, exploiting the admissibility of the structural rules.

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$$\frac{a \Vdash^{\forall} \bigvee_{k>0} A_k, \Gamma \Rightarrow \Delta}{a \Vdash^{\forall} A_k, \Gamma \Rightarrow \Delta}\ Mon_{\bigvee} \qquad \frac{m(a) \Vdash^{\forall} \bigwedge_{k>0} A_k, \Gamma \Rightarrow \Delta}{a \Vdash^{\forall} \&_{k>0} A_k, \Gamma \Rightarrow \Delta}\ Mon_{\bigwedge}$$

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$$\frac{a \Vdash^{\forall} \bigvee_{k>0} A_k, \Gamma \Rightarrow \Delta}{a \Vdash^{\forall} A_k, \Gamma \Rightarrow \Delta}\, Mon_\bigvee \qquad \frac{m(a) \Vdash^{\forall} \bigwedge_{k>0} A_k, \Gamma \Rightarrow \Delta}{a \Vdash^{\forall} \mathop{\&}_{k>0} A_k, \Gamma \Rightarrow \Delta}\, Mon_\bigwedge$$

Proof. By induction on the height of derivations, exploiting the admissibility of the structural rules.

Theorem. For every $A \in FOR$, the rule

$$\dfrac{x : A, a \in N(x), a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta}{x : A, \Gamma \Rightarrow \Delta}\; Mon^{+},\ a \text{ fresh}$$

is admissible in $G3I_\omega$.

Proof. By transfinite induction on the complexity of A. For example, if $A \equiv B \lor C$, we proceed as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{x : B \lor C, x : B, a \in N(x), a \Vdash^{\forall} B \lor C, \Gamma \Rightarrow \Delta}{x : B \lor C, x : B, a \in N(x), a \Vdash^{\forall} B, \Gamma \Rightarrow \Delta}\; Mon_{\lor}}{x : B \lor C, x : B, \Gamma \Rightarrow \Delta}\; IH \qquad \dfrac{\dfrac{x : B \lor C, x : C, b \in N(x), b \Vdash^{\forall} B \lor C, \Gamma \Rightarrow \Delta}{x : B \lor C, x : C, b \in N(x), b \Vdash^{\forall} C, \Gamma \Rightarrow \Delta}\; Mo}{x : B \lor C, x : C, \Gamma \Rightarrow \Delta}\; IH}{x : B \lor C, x : B \lor C, \Gamma \Rightarrow \Delta}\; L\lor}{x : B \lor C, \Gamma \Rightarrow \Delta}\; Ctr$$

Caveat. The restriction to *FOR* instead of the full language of the labelled calculus is crucial. If A were $B \supset C$ the rule Mon^{+} would be unsound.

An example of a derivation

We give an example of a derivation: $\vdash_{G3I_\omega} \Rightarrow x : A \to (B \to A)$.

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\cfrac{a \in N(x), y \in a, z \in b, b \in N(y), b \Vdash^{\forall} A, y : A, z : B, z : A \Rightarrow x : A \to (B \to A), y : B \to A, z : A}
{a \in N(x), y \in a, z \in b, b \in N(y), b \Vdash^{\forall} A, y : A, z : B \Rightarrow x : A \to (B \to A), y : B \to A, z : A}\; L\Vdash
}{a \in N(x), y \in a, z \in b, b \in N(y), b \Vdash^{\forall} A, y : A \Rightarrow x : A \to (B \to A), y : B \to A, z : B \supset A}\; R\supset
}{a \in N(x), y \in a, b \in N(y), b \Vdash^{\forall} A, y : A \Rightarrow x : A \to (B \to A), y : B \to A, b \Vdash^{\forall} B \supset A}\; R\Vdash^{\forall}
}{a \in N(x), y \in a, b \in N(y), b \Vdash^{\forall} A, y : A \Rightarrow x : A \to (B \to A), y : B \to A}\; R\to
}{a \in N(x), y \in a, y : A \Rightarrow x : A \to (B \to A), y : B \to A}\; Mon^{+}
}{a \in N(x), y \in a \Rightarrow x : A \to (B \to A), y : A \supset (B \to A)}\; R\supset
}{a \in N(x) \Rightarrow x : A \to (B \to A), a \Vdash^{\forall} A \supset (B \to A)}\; R\Vdash^{\forall}
}{a \in N(x) \Rightarrow x : A \to (B \to A)}\; R\to
}{\Rightarrow x : A \to (B \to A)}\; Nondeg
}{}
$$

The topsequent is derivable.

Theorem. In

$$\frac{\Gamma \Rightarrow \Delta, \varphi \quad \varphi, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'} \; \mathit{Cut}$$

if the premisses have cut-free derivations then so has the conclusion.

Proof. By double transfinite induction on lexicographically ordered pairs with main induction hypothesis on the weight of φ and secondary induction hypothesis on the natural sum of height of derivations of the premises of the cut. As usual, we distinguish between:

1. Cuts with cut formula principal in both premisses
2. Cuts with cut formula non-principal in at least one premiss

Admissibility of cut for $\mathbf{G3I}_\omega$ (cont.)

If the cut formula $x : \bigwedge_{n>0} A_n$ is **principal in both premisses**, we have

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ a \in N(x), \Gamma \Rightarrow \Delta, x : \bigwedge_{k>0} A_k, a \Vdash^{\forall} \bigwedge_{k>0} A_k\end{array}}{a \in N(x), \Gamma \Rightarrow \Delta, x : \bigwedge_{k>0} A_k}\, R\bigwedge_k \qquad \dfrac{\begin{array}{c}\vdots\\ b \in N(x), b \Vdash^{\forall} \&_{k>0} A_k, \Gamma' \Rightarrow \Delta'\end{array}}{x : \bigwedge_{n>0}, \Gamma' \Rightarrow \Delta'}\, L\bigwedge, b \textit{ fresh}}{a \in N(x), \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\, \textit{Cut}$$

which we transform into

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ a \in N(x), \Gamma \Rightarrow \Delta, x : \bigwedge_{k>0} A_k, a \Vdash^{\forall} \underset{k>0}{\&} A_k\end{array} \qquad \dfrac{\begin{array}{c}\vdots\\ b \in N(x), b \Vdash^{\forall} \underset{k>0}{\&} A_k, A_n, \Gamma' \Rightarrow \Delta'\end{array}}{x : \bigwedge_{k>0} A_k, \Gamma' \Rightarrow \Delta'}\, L\bigwedge}{a \in N(x), \Gamma, \Gamma' \Rightarrow \Delta, \Delta', a \Vdash^{\forall} \underset{k>0}{\&} A_k}\, \textit{Cut} \qquad \dfrac{b \in N(x), b \Vdash^{\forall} \underset{k>0}{\&} A_k, \Gamma' \Rightarrow \Delta'}{a \in N(x), a \Vdash^{\forall} \underset{k>0}{\&} A_k, \Gamma' \Rightarrow \Delta'}\, (b/a)}{a \in N(x), a \in N(x), \Gamma, \Gamma', \Gamma' \Rightarrow \Delta, \Delta', \Delta'}\, \textit{Cut}}{a \in N(x), \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\, \textit{Ctr}^*$$

The rank of both cuts is reduced: first cut has lower height, second has lower depth, so the induction hypothesis applies to both
The contractions are admissible.

Completeness of $\mathbf{G3I}_\omega$

Definition

Given a set S of world labels x and a set NL of neighborhood labels a, and a neighborhood model $\mathcal{M} = \langle W, N, v \rangle$, an *SN* realisation (ρ, σ) is a pair of functions s.t. $\rho(x) \in W$ and $\sigma(a) \in N(w)$ for some $w \in W$. An *SN*-realisation (ρ, σ) has to respect formal intersection of the language and the operation m. $\mathcal{M}$ *satisfies a formula* ϕ *under an SN-realisation* (σ, ρ), $\mathcal{M} \vDash_{\rho,\sigma} \phi$, where we assume that the labels in A occur in S, NL. The definition extends the usual cases for the propositional connectives by cases on the form of ϕ:

- $\mathcal{M} \vDash_{\rho,\sigma} x \in a$ if $\rho(x) \in \sigma(a)$
- $\mathcal{M} \vDash_{\rho,\sigma} a \in N(x)$ if $\sigma(a) \in N(\rho(x))$
- $\mathcal{M} \vDash_{\rho,\sigma} x : A$ if $\rho(x) \vDash A$
- $\mathcal{M} \vDash_{\rho,\sigma} a \Vdash^{\forall} A$ if for all y in $\sigma(a)$, $\rho(y) \vDash A$
- $\mathcal{M} \vDash_{\rho,\sigma} x : A \to B$ if for some a, $\sigma(a) \in N(\rho(x))$ and $\sigma(a) \subseteq [A \supset B]$
- $\mathcal{M} \vDash_{\rho,\sigma} x : \bigwedge\limits_{k>0} A_k$ if for some a, $\sigma(a) \in N(\rho(x))$ and $\sigma(a) \subseteq [\mathop{\&}\limits_{k>0} A_k]$

where $[A] = \{x \in W \mid x \vDash A\}$. Given a sequent $\Gamma \Rightarrow \Delta$, $\mathcal{M} \vDash_{\rho,\sigma} \Gamma \Rightarrow \Delta$ holds if, whenever $\mathcal{M} \vDash_{\rho,\sigma} \phi$ for all formulas $\phi \in \Gamma$, then $\mathcal{M} \vDash_{\rho,\sigma} \psi$ for some formula $\psi \in \Delta$. We further define $\mathcal{M}$-validity by:

$$\mathcal{M} \vDash \Gamma \Rightarrow \Delta \text{ iff } \mathcal{M} \vDash_{\rho,\sigma} \Gamma \Rightarrow \Delta \text{ for every } SN\text{-realisation } (\rho, \sigma).$$

A sequent $\Gamma \Rightarrow \Delta$ is valid in a neighborhood frame if $\mathcal{M} \vDash \Gamma \Rightarrow \Delta$ for every neighborhood model based on it.

Soundness and completeness (cont.)

Theorem. If $\Gamma \Rightarrow \Delta$ is derivable in $G3I_\omega$, then it is valid in the neighborhood semantics for infinitary intuitionistic logic.

Proof. By induction on the height of derivation.

Theorem. If $\vDash A$, then $G3I_\omega \vdash \Rightarrow x : A$.

Proof. An indirect strategy consists in showing that the unlabelled calculus $G3i_\omega$ embeds into $G3I_\omega$. For a more direct strategy we can define a procedure to construct a reduction tree and we extract an infinite countermodel from a non-terminating branch in a failed proof search (the proof is non-constructive, because we make essential use of König's lemma).

Infinitary $S4$ modal logic - Syntax

Infinitary modal logic is modal logic extended with countable conjunctions and disjunctions. We consider the following axiomatisation of $S4_\omega$:

Axioms

The axioms of the classical logic, plus:

C1. $\bigwedge_{k>0} A_k \to A_k$

C2. $A_k \to \bigvee_{k>0} A_k$

C3. $\bigwedge_{k>0} (A \to B_k) \to (A \to \bigwedge_{k>0} B_k)$

C4. $\bigwedge_{k>0} (A_k \to B) \to (\bigvee_{k>0} A_k \to B)$

4 $\Box A \to \Box\Box A$

T $\Box A \to A$

K $\Box(A \to B) \to (\Box A \to \Box B)$

Inference Rules

$$\frac{\Gamma' \vdash A \qquad \Gamma'' \vdash A \to B}{\Gamma', \Gamma'' \vdash B}\ \mathrm{MP} \qquad \frac{\vdash A}{\vdash \Box A}\ RN$$

$$\frac{\{\Gamma \vdash A_k\}_{k>0}}{\Gamma \vdash \bigwedge_{k>0} A_k}\ Adj$$

Infinitary $S4$ modal logic - Semantics

A $S4$ neighborhood model for infinitary modal logic is a triple $\mathcal{M} = \langle W, N, v \rangle$ where $\langle W, N \rangle$ is an intuitionistic neighborhood frame and $v : AT \to P(W)$ is a valuation function. For every world $x \in W$ and every formula $A \in FOR^{\Box}_{\omega}$ the satisfiability condition in the model $\mathcal{M}$, $x \vDash A$, is defined inductively as follows:

- $x \vDash p$ iff $x \in v(p)$. Classical connectives as usual.
- $x \vDash \bigvee_{k>0} B_k$ iff $x \vDash B_k$ for some $k > 0$.
- $x \vDash \bigwedge_{k>0} B_k$ iff $x \vDash B_k$ for every $k > 0$.
- $x \vDash \Box B$ iff $\exists a \in N(x)(a \Vdash^{\forall} B)$.

Validity is defined as usual.

Theorem. $S4 \vdash A$ iff $\vDash_{S4} A$.

Proof. From left to right we proceed by induction on the height of derivations in the axiomatic calculus $S4_{\omega}$. From right to left the strategy is quite similar to the standard one detailed in Pacuit (2017), we make use once again of the notion of environment of a subset of formulas in order to carry out the Lindenbaum construction.

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Labelled sequent calculus $G3S4_\omega$

Initial Sequents

$$\dfrac{}{x : p, \Gamma \Rightarrow \Delta, x : p}\ Ax \qquad \dfrac{}{x : \bot, \Gamma \Rightarrow \Delta}\ L\bot$$

Logical Rules

$$\dfrac{\Gamma \Rightarrow \Delta, x : A \qquad x : B, \Gamma \Rightarrow \Delta}{x : A \to B, \Gamma \Rightarrow \Delta}\ L\to \qquad \dfrac{x : A, \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : A \to B}\ R\to$$

$$\dfrac{x : \bigwedge_{k>0} A_k, x : A_k, \Gamma \Rightarrow \Delta}{x : \bigwedge_{k>0} A_k, \Gamma \Rightarrow \Delta}\ L\bigwedge \qquad \dfrac{\{\Gamma \Rightarrow \Delta, x : A_k \mid k > 0\}}{\Gamma \Rightarrow \Delta, x : \bigwedge_{k>0} A_k}\ R\bigwedge$$

$$\dfrac{\{x : A_k, \Gamma \Rightarrow \Delta \mid k > 0\}}{x : \bigvee_{k>0} A_k, \Gamma \Rightarrow \Delta}\ L\bigvee \qquad \dfrac{\Gamma \Rightarrow \Delta, x : \bigvee_{k>0} A_k, x : A_k}{\Gamma \Rightarrow \Delta, x : \bigvee_{k>0} A_k}\ R\bigvee$$

$$\dfrac{a \in N(x), a \Vdash^\forall A, \Gamma \Rightarrow \Delta}{x : \Box A, \Gamma \Rightarrow \Delta}\ L\Box\text{, } a \text{ fresh} \qquad \dfrac{a \in N(x), \Gamma \Rightarrow \Delta, x : \Box A, a \Vdash^\forall A}{a \in N(x), \Gamma \Rightarrow \Delta, x : \Box A}\ R\Box$$

$G3S4_\omega$ is obtained adding to the above rules the standard ones for $\wedge, \vee$, the auxiliary rules of $G3I_\omega$ with the exception of *Mon* and rules $L \Vdash^\forall$ and $R \Vdash^\forall$. Every rule is invertible, weakening and contraction are admissible. Cut is admissible.

Gödel translation

The following is a modification of Gödel original translation. The function $t : FOR \to FOR^{\Box}$ where $FOR^{\Box}$ denotes the language of propositional modal logic, is defined as follows:

- $(p)^t = \Box p$
- $(F \# G)^t = F^t \# G^t$, $\# \in \{\wedge, \vee\}$
- $(F \to G)^t = \Box(F^t \to G^t)$

is the modal translation from I to S4.

The natural extension of the translation to the infinitary setting would consist in $(\bigwedge\limits_{k>0} A_k)^t = \bigwedge\limits_{k>0} A_k^t$ and $(\bigvee\limits_{k>0} A_k)^t = \bigvee\limits_{k>0} A_k^t$.

Fact. The natural extension of the translation is not faithful. Consider $\bigwedge\limits_{k>0} (p_k \vee q) \to \bigwedge\limits_{k>0} p_k \vee q$. Such formula is not provable in intuitionistic infinitary logic, yet its translation is provable in $G3S4_{\omega}$.

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We consider the following extension of Gödel embedding:

$$(\bigvee_{k>0} A_k)^t = \bigvee_{k>0} A_k^t \text{ and } (\bigwedge_{k>0} A_k)^t = \Box \bigwedge_{k>0} A_k^t$$

We extend t to the full language of the labelled calculus $\mathbf{G3I}_\omega$.
$(A \supset B)^t = A^t \to B^t$, $(\underset{k>0}{\&} B_k)^t = \bigwedge_{k>0} B_k^t$ and:

- $(a \Vdash^\forall B)^t = a \Vdash^\forall B^t$
- $(x : B)^t = x : B^t$
- $(x \in a)^t = x \in a$
- $(a \in N(x))^t = a \in N(x)$

Given a multiset of labelled formulas $\Gamma = \varphi_1, ..., \varphi_n$, Γ^t denotes $\varphi_1^t, ..., \varphi_n^t$

A proof-theoretical proof of the embedding - Soundness

Theorem. Let Γ, Δ be multisets of labelled formulas: if $G3I_\omega \vdash \Gamma \Rightarrow \Delta$, then $G3S4_\omega \vdash \Gamma^t \Rightarrow \Delta^t$.

Proof. The proof is by transfinite induction on the height of derivation in the calculus $G3I_\omega$ distinguishing cases according to the last rule applied. We discuss one case.
If $n > 0$ and the last rule is *Mon* we have the following situation:

$$\frac{x : p, a \in N(x), a \Vdash^{\forall} p, \Gamma \Rightarrow \Delta}{x : p, \Gamma \Rightarrow \Delta}\ \textit{Mon}, a \text{ fresh}$$

The induction hypothesis yields a derivation of
$x : \Box p, a \in N(x), a \Vdash^{\forall} \Box p, \Gamma^t \Rightarrow \Delta^t$. We construct the following derivation:

$$\dfrac{\dfrac{\dfrac{x : \Box p \Rightarrow x : \Box\Box p \qquad \dfrac{x : \Box p, a \in N(x), a \Vdash^{\forall} \Box p, \Gamma^t \Rightarrow \Delta^t}{x : \Box p, x : \Box\Box p, \Gamma^t \Rightarrow \Delta^t}\ L\Box}{x : \Box p, x : \Box p, \Gamma^t \Rightarrow \Delta^t}\ \textit{Cut}}{x : \Box p, \Gamma^t \Rightarrow \Delta^t}\ \textit{Ctr}}{}$$

Where the topmost sequent $x : \Box p \Rightarrow x : \Box\Box p$ on the left is derivable in $G3S4_\omega$.

A proof-theoretical proof of the embedding - Faithfulness

The following is a generalization to the infinitary case of Dyckhoff, Negri (2012).

Lemma. Let Γ, Δ be multisets of labelled formulas, Γ' a multiset of atomic formulas and of labelled formulas of the form $a \Vdash^{\forall} p$, Δ' a multiset of atomic formulas, Ω a multiset of relational atoms:

$$\text{if } G3S4_{\omega} \vdash \Omega, \Gamma^{t}, \Gamma' \Rightarrow \Delta^{t}, \Delta', \text{ then } G3I_{\omega} \vdash \Omega, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'$$

Proof. By induction on the height of derivations distinguishing cases according to the last rule applied. For example:

If $n > 0$ and the last rule is $R\Box$ and the principal formula is $x : \Box p$ we have:

$$\frac{a \in N(x), \Omega, \Gamma^{t}, \Gamma' \Rightarrow \Delta^{t}, x : \Box p, a \Vdash^{\forall} p, \Delta'}{a \in N(x), \Omega, \Gamma^{t}, \Gamma' \Rightarrow \Delta^{t}, x : \Box p, \Delta'} \; R\Box$$

By height-preserving invertibility of $R \Vdash^{\forall}$ we obtain $a \in N(x), x \in a, \Omega, \Gamma^{t}, \Gamma' \Rightarrow \Delta^{t}, x : \Box p, x : p, \Delta'$. Induction hypothesis yields $a \in N(x), x \in a, \Omega, \Gamma, \Gamma' \Rightarrow \Delta, x : p, x : p, \Delta'$ and by admissibility of contraction and *Ref* we obtain the desired conclusion.

Corollary G3I$_{\omega}$ $\vdash \Rightarrow x : A$ iff **G3S4$_{\omega}$** $\vdash \Rightarrow x : A^{t}$.
Proof. From left to right we proceed by induction on the height of derivations, from right to left we exploit the Lemma.

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Corollary $\mathbf{G3I}_{\omega} \vdash \Rightarrow x : A$ iff $\mathbf{G3S4}_{\omega} \vdash \Rightarrow x : A^t$.
Proof. From left to right we proceed by induction on the height of derivations, from right to left we exploit the Lemma.

Conclusion

We have introduced a neighborhood semantics for infinitary intutionistic logic and we have extracted a labelled sequent calculus from it. The calculus has good structural properties and allows for a direct proof of completeness. We have also extended the Gödel-McKinsey-Tarski embedding to the infinitary languages.

Further work includes the extension to first order intuitionistic logic and the study of neighborhood semantics for intermediate logics.

With respect to modal logics it would be interesting to see whether the approach of labelled sequent calculi based on neighborhood semantics might be applied to provability logics such as *GL* and *Grz*.

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