Modal operators and toric ideals

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Kripke frames

Let $\mathfrak{P}$ be the set of modal formulas built starting from a given set of propositional letters, by repeated application of $\neg, \wedge, \Box$.

Definition

- A *Kripke frame* is a pair $\mathcal{K} = (W, \mathcal{E})$, where W is a non-empty set and $\mathcal{E}$ is a binary relation on W
- $\mathcal{K}$ is *locally finite* if for every $w \in W$ the set $\{w' \in W \mid w\mathcal{E}w'\}$ is finite
- A *subframe* of $\mathcal{K}$ is a Kripke frame $\mathcal{K}' = (W', \mathcal{E}')$ such that $W' \subseteq W, \mathcal{E}' = \mathcal{E} \cap W'^2$
- If $w\mathcal{E}w'$, then w' is *accessible* from w
- $N(w) = \{w' \in W \mid w\mathcal{E}w'\}$ is the *neighbourhood* of w
- The *incidence matrix* E of $\mathcal{K}$ is defined by $E(w, w') = 1 \Leftrightarrow w\mathcal{E}w'$

Kripke models

Definition

A *Kripke model* $\mathcal{K}_\Phi = (W, \mathcal{E}, \Phi)$ is a Kripke frame $\mathcal{K} = (W, \mathcal{E})$ endowed with a function

$$\Phi : \mathfrak{P} \times W \to 2 = \{0, 1\}$$

such that:

- $\Phi(\neg p, w) = 1 - \Phi(p, w)$
- $\Phi(p \wedge q, w) = \Phi(p, w)\Phi(q, w)$
- $\Phi(\Box p, w) = \prod_{(w,w') \in \mathcal{E}} \Phi(p, w')$

Function Φ can be extended to formulas built using the remaining logical symbol $\vee, \to, \leftrightarrow, \Diamond$, using the defined meaning of these symbols.

Definition

- $(\mathcal{K}_\Phi, w) \Vdash p$ means $\Phi(p, w) = 1$
- $\mathcal{K}_\Phi \Vdash p$ means $(\mathcal{K}_\Phi, w) \Vdash p$ for every $w \in W$
- $\mathcal{K} \Vdash p$ means $\mathcal{K}_\Phi \Vdash p$ for every Φ

Formulas as functions

Fix a Kripke model $\mathcal{K}_\Phi$. Then to every $p \in \mathfrak{P}$ provides a function $\Phi(p, \cdot)$:

$$\begin{array}{rcl} W & \to & 2 \\ w & \mapsto & \Phi(p, w) \end{array}$$

It is the characteristic function of $\{w \in W \mid \Phi(p, w) = 1\}$, the truth set of p.
If the model $\mathcal{K}_\Phi$ is understood, such a function can be denoted by the same name as the formula p. Therefore $p \in 2^W$.

Remark. Formulas p, q determine the same function if and only if $\mathcal{K}_\Phi \Vdash p \leftrightarrow q$.

The monoid 2^W

$(2^W, \cdot)$ is a monoid under pointwise multiplication. It is isomorphic to $(\mathcal{P}(W), \cap)$.

Let $\mathbb{C}^W$ be the monoid of complex-valued functions on W, under pointwise multiplication. Then $2^W \subseteq \mathbb{C}^W$, namely

$$2^W = \{a \in \mathbb{C}^W \mid a^2 = a\}$$

Example

Given a formula $p \in 2^W$, the fomula $\Box p$ is true in a node w if and only if p is true in any node accessible from w. That is:

$$\Box p(w) = \prod_{w' \in N(w)} p(w') \tag{1}$$

Note that equation (1) defines a function $\Box : 2^W \to 2^W$.
If $\mathcal{K}$ is locally finite, equation (1) extends to a function $\Box : \mathbb{C}^W \to \mathbb{C}^W$.

Proposition

$\Box : 2^W \to 2^W$ is a homomorphism.
If $\mathcal{K}$ is locally finite, then $\Box : \mathbb{C}^W \to \mathbb{C}^W$ is a homomorphism.

Proof. $\Box 1 = 1$ and $\Box(a \cdot b) = \Box a \cdot \Box b$.

Cycles and lines

Definition

Let $\mathcal{K} = (W, \mathcal{E})$ be a Kripke frame.

- A *cycle* in $\mathcal{K}$ is a subframe $(\{x_0, \ldots, x_n\}, \mathcal{E}')$, for some $n \geq 0$, such that $x_0 \mathcal{E}' \ldots \mathcal{E}' x_n \mathcal{E}' x_0$ and the relation $\mathcal{E}'$ does not hold for any other pair of elements of $\{x_0, \ldots, x_n\}$
- A *line* in $\mathcal{K}$ is a subframe $(\{x_i\}_{i \in \mathbb{Z}}, \mathcal{E}')$ such that $\forall i, j \in \mathbb{Z}\ (x_i \mathcal{E}' x_j \Leftrightarrow j = i + 1)$

Remark. The requirement that a cycle is a subframe makes this definition stronger than the usual one: the only edges of $\mathcal{K}$ between the elements of the cycle are the edges of the cycle.

$\Box$ as an isomorphism

Theorem

Let $\mathcal{K} = (W, \mathcal{E})$ be a Kripke frame. Then $\Box : 2^W \to 2^W$ is an isomorphism if and only if $\mathcal{K}$ is the disjoint union of its cycles and lines. That is, if $\{(W_i, \mathcal{E}_i)\}_{i \in I}$ is the collection of all cycles and lines in $\mathcal{K}$:

- $\{W_i\}_{i \in I}$ is a partition of W and
- $\{\mathcal{E}_i\}_{i \in I}$ is a partition of $\mathcal{E}$

Corollary

Let $\mathcal{K} = (W, \mathcal{E})$ be a finite Kripke frame. Then $\Box : 2^W \to 2^W$ is an isomorphism if and only if it is injective, if and only if it is surjective, if and only if $\mathcal{K}$ is the disjoint union of its cycles.

$\Box$ as an isomorphism

In fact, the following holds.

Proposition

- If every $b \in 2^W$ assuming exactly once the value 0 is in the range of $\Box$, then $\Box : 2^W \to 2^W$ is surjective
- If $\Box a \neq \Box a'$ for every $a, a' \in 2^W$ differing on exactly one element of W, then $\Box : 2^W \to 2^W$ is injective

Corollary

Let $\mathcal{K}$ be a finite Kripke frame. Then the following are equivalent:

1. Every $b \in 2^W$ assuming exactly once the value 0 is in the range of $\Box$
2. If $a, a' \in 2^W$ differ on exactly one element of W then $\Box a \neq \Box a'$
3. $\Box : 2^W \to 2^W$ is an isomorphism

range(□), algebraically

Let $\mathcal{K} = (W, \mathcal{E})$ be a finite Kripke frame, with $W = \{1, \ldots, K\}$.

Problem

Given a Kripke frame $\mathcal{K}$, characterise the formulas $\Box a$.

That is, characterise $\mathrm{range}(\Box)$.

There is an algebraic approach to this question.

$$\mathrm{range}(\Box) \subseteq 2^W \subseteq \mathbb{C}^W$$

Therefore $\mathrm{range}(\Box)$, being finite, is a subvariety of $\mathbb{C}^K$. We describe a procedure to find its equations.

Note. $(2, +, \cdot)$ too is a field, therefore $\mathrm{range}(\Box)$ is a subvariety of 2^K. However the use of the field $\mathbb{C}$ allows to apply known results in commutative algebra.
I do not know if all the arguments can be carried out directly in 2^K.

Some basic definitions and facts

- If I is an ideal in the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$, the *variety* of I is

$$V(I) = \{a \in \mathbb{C}^n \mid \forall f \in I \ f(a) = 0\}$$

- If $A \subseteq \mathbb{C}^n$, the *ideal* of A is

$$I(A) = \{f \in \mathbb{C}[x_1, \ldots, x_n] \mid \forall a \in A \ f(a) = 0\}$$

- Every ideal I in $\mathbb{C}[x_1, \ldots, x_n]$ is finitely generated, therefore $I = \langle f_1, \ldots, f_r \rangle$ for some polynomials $f_1, \ldots, f_r$

Equations for $\mathrm{range}(\Box)$

Recall that $W = \{1, \ldots, K\}$, and $\Box$ is defined by

$$\Box a(w) = \prod_{w' \in N(w)} a(w') = \prod_{w' \in W} (a(w'))^{E(w,w')} \tag{2}$$

Consider two sequences of indeterminates:

$$t_w = a(w), \quad z_w = \Box a(w), \qquad w \in W$$

in the polynomial ring $\mathbb{C}[t_1, \ldots, t_K, z_1, \ldots, z_K]$.
In the coordinates $(t_1, \ldots, t_K, z_1, \ldots, z_K)$ the conditions on the pairs $(a, \Box a)$, for $a \in 2^W$ are:

- $t_w^2 - t_w = 0$ for $w \in W$, since $a(w) \in 2$. The corresponding ideal is

$$\mathcal{I}_L = \langle t_w^2 - t_w \mid w \in W \rangle$$

- $z_w - \prod_{w' \in N(w)} t'_w = 0$, by (2). The corresponding ideal is

$$\mathcal{I}_T = \langle z_w - \prod_{w' \in N(w)} t_{w'} \mid w \in W \rangle$$

Equations for $\mathrm{range}(\Box)$

$\mathcal{I}_T$ is a toric ideal in the indeterminates z_w. Toric ideals are special binomial ideals; they are applied for instance in algebraic statistics for contingency tables, to describe varieties (that is, statistical models) for finite sample spaces.

Let

$$\mathcal{I} = \mathcal{I}_L + \mathcal{I}_T$$

In the affine space $\mathbb{C}^{2K} = \mathbb{C}^K_{(t)} \times \mathbb{C}^K_{(z)}$ are contained the affine varieties of the three ideals:

$$V(\mathcal{I}_L), \quad V(\mathcal{I}_T), \quad V(\mathcal{I})$$

- $V(\mathcal{I}_L) = 2^K \times \mathbb{C}^K$
- $V(\mathcal{I}_T)$ is the toric variety of the adjacency matrix E of the Kripke frame.

Equations for $\mathrm{range}(\Box)$

Projecting these three varieties on $\mathbb{C}^K_{(z)}$, let:

$$\tilde{V}(\mathcal{I}_L) = \pi_{\mathbb{C}^K_{(z)}}(V(\mathcal{I}_L)), \quad \tilde{V}(\mathcal{I}_T) = \pi_{\mathbb{C}^K_{(z)}}(V(\mathcal{I}_T)), \quad \tilde{V}(\mathcal{I}) = \pi_{\mathbb{C}^K_{(z)}}(V(\mathcal{I}))$$

Then

$$\mathrm{range}(\Box) = \tilde{V}(\mathcal{I})$$

On the other hand, let the elimination ideals of the indeterminates t_w be:

$$\widetilde{\mathcal{I}_L} = Elim(t_1, \ldots, t_K, \mathcal{I}_L)$$

$$\widetilde{\mathcal{I}_T} = Elim(t_1, \ldots, t_K, \mathcal{I}_T)$$

$$\tilde{\mathcal{I}} = Elim(t_1, \ldots, t_K, \mathcal{I})$$

Fact. The varieties (in $\mathbb{C}^K_{(z)}$) of these elimination ideals are the Zariski closures of the above projection varieties.

Since $\mathrm{range}(\Box)$ is finite, it is Zariski closed. Therefore

$$\mathrm{range}(\Box) = \tilde{V}(\mathcal{I}) = V(\tilde{\mathcal{I}})$$

Equations for $\mathrm{range}(\Box)$

Therefore:

- Any set of generators of $\tilde{\mathcal{I}}$ provides a system of equations for $\mathrm{range}(\Box)$.
- $\tilde{\mathcal{I}}$ is both an elimination ideal and a binomial ideal. A set of generators can be computed through Gröbner bases with symbolic software.

Notation.

- Given $\beta \in \mathbb{N}^K$, let $z^\beta = z_1^{\beta_1} \cdot \ldots \cdot z_K^{\beta_K}$
- Given $\alpha \in \mathbb{Z}^K$, let $\alpha_+, \alpha_- \in \mathbb{N}^K$ have disjoint support and be such that $\alpha = \alpha_+ - \alpha_-$

Theorem

1. The ideal $\widetilde{\mathcal{I}_T}$ is generated by the binomials

$$z^{\alpha_+} - z^{\alpha_-}, \qquad \text{for } \alpha \in \mathbb{Z}^K \cap ker(E^t)$$

2. The ideal $\tilde{\mathcal{I}}$ is generated by the binomials

$$z_w^2 - z_w, \qquad \text{for } w \in W$$

plus the square-free binomials of the form

$$z^u - z^v$$

for $u, v \in 2^K$ such that $supp(E^t u) = supp(E^t v)$.

Tame frames

The binomials $z^{\alpha_+} - z^{\alpha_-}$ for $\alpha \in \mathbb{Z}^K \cap ker(E^t)$, that appear as generators of $\widetilde{\mathcal{I}_T}$, are easier to calculate than the binomials $z^u - z^v$ for $u, v \in 2^K$ and $supp(E^t u) = supp(E^t v)$, that are among the generators of $\tilde{\mathcal{I}}$.

In fact

$$J = \langle z_w^2 - z_w \mid w \in W \rangle + \widetilde{\mathcal{I}_T} \subseteq \tilde{\mathcal{I}}$$

A desirable situation would be when $\mathrm{range}(\Box) = V(J)$. In general this equality does not hold. Notice that

$$\mathrm{range}(\Box) = V(\tilde{\mathcal{I}}) \subseteq V(J)$$

Definition

The Kripke frame $\mathcal{K}$ is *tame* if $\mathrm{range}(\Box) = V(J)$.

Theorem

$\mathcal{K}$ is tame if and only if $J = \tilde{\mathcal{I}}$

Partitioning frames

Definition

The Kripke frame $\mathcal{K}$ is a *partitioning frame* if

$$\forall w, w' \in W \; (N(w) \cap N(w') \neq \emptyset \Rightarrow N(w) = N(w'))$$

Theorem

Finite partitioning frames are tame.

Examples

The following are partitioning frames:

- Complete bipartite graphs.
- Graphs for which $\Box$ is an isomorphism. Here $\widetilde{\mathcal{I}_T}$ is the null ideal, since E is non-singular. Moreover $\tilde{\mathcal{I}} = \langle z_w^2 - z_w \mid w \in W \rangle$, since in every row and every column of E there is exactly one entry 1, therefore $\forall u, v \in 2^W \ (supp(E^t u) = supp(E^t v) \Leftrightarrow u = v)$.
- Equivalence relations. These are the Kripke frames defined by epistemic logic $S5$, that is, by the axioms $\Box p \to p$ and $\Diamond p \to \Box \Diamond p$.
- Directed rooted trees and directed trees with inversed arrows.

Questions

1. Is there a set of axioms for tame frames, that is a set of axioms whose finite models are exactly the tame frames?
2. Are there other classes of Kripke frames for which the set of equations for $\mathrm{range}(\Box)$ can be simplified?
3. I presented a procedure for computing the equations of $\mathrm{range}(\Box)$, possibly by using symbolic software. Who cares?