CLASS FORCING AND SET-THEORETIC GEOLOGIES

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Introduction

- The technique of forcing is customarily thought of as a method for constructing *outer* as opposed to *inner* models of set theory.
- A switch in perspective, however, allows us to view forcing as a method of describing inner models as well.
- The idea is simply to search inwardly for how the universe V might itself have arisen by forcing.

Definition

A transitive class $W \subset V$ is called a ground of V if $W \models \mathrm{ZFC}$ and there exists a p.o. $P \in W$ such that $V = W[G]$ for some G which is P-generic over W.

Set-Theoretic Geology

This change in viewpoint is the basis for a collection of questions leading to the topic we refer to as **set-theoretic geology**.

Questions

- Are there any inner models $W \subset V$ such that V is a set-forcing extension of W?
- How many are they?
- Is the intersection of two grounds a ground?

Set forcing

Forcing is a method for expanding a given model of ZFC. Given a c.t.m. M of ZFC, forcing enables us to produce a new transitive model $M[G]$ such that

- $M \subset M[G]$ and $G \in M[G]$,
- $ORD^M = ORD^{M[G]}$,
- $M[G] \vDash$ ZFC.

The model M has a surprising degree of access to the objects and truths of $M[G]$.

Theorem (Definability lemma)

For any φ, the relation "$p \Vdash \varphi(\tau_1, \ldots, \tau_n)$" of $p \in P$, $\tau_1, \ldots, \tau_n \in M^P$ is definable in M.

Theorem (Truth lemma)

If $\varphi(x_1, \ldots, x_n)$ is a formula, G is P-generic over M and $\tau_1, \ldots, \tau_n$ are P-names, then

$$M[G] \vDash \varphi(i_G(\tau_1), \ldots, i_G(\tau_n)) \text{ if and only if } \exists p \in G(p \Vdash \varphi(\tau_1, \ldots, \tau_n)).$$

The Ground Axiom: the universe was not obtained by forcing

We investigate the idea of "undoing forcing".

Question (Changing the viewpoint)

Is there a nontrivial ground of V?

Definition

The Ground Axiom (GA) asserts that the universe V is not a set forcing extension of any **proper** inner model. That is, if W is a ground of V, then $V = W$.

Example

The constructible universe L satisfies the Ground Axiom.

What can we say about GA?

We want to discuss the following:

Question 1

Does the Ground Axiom implies the Generalized Continuum Hypothesis?

After all, the only way we know how to violate GCH is by forcing, and under GA the universe is not a forcing extension, so an affirmative answer seems reasonable.

Question 2

If M satisfies ZFC, is there a forcing extension of M which satisfies ZFC+GA?

Ground model definability

Remark

Although the formulation quantifies over inner models, the Ground Axiom is actually first-order expressible in the language of set theory.

The fact that GA is first order expressible is strongly related with a theorem of Laver (2004), which answers a fundamental question about forcing.

Question

Is the model $M \models \mathrm{ZFC}$ definable in its set-forcing extensions $M[G]$?

It turns out that it is.

Theorem (Laver, Woodin)

Every ground of V is definable by some first-order formula.

Uniform definability of grounds

There is a formula $\phi(z, x)$ such that if W is a ground of V, then there is $r \in W$ such that

$$x \in W \text{ if and only if } V \models \phi(r, x).$$

Working a little bit we get more.
Indeed, all grounds can be defined by some **uniform** way.

Theorem (uniform definability)

There is a formula $\Psi(y, x)$ such that

1. For each $r \in V$, the class $W_r = \{x : \Psi(r, x)\}$ is a ground of V and $r \in W_r$.
2. For every ground W of V, there is a parameter $r \in W$ such that $W = \{x : \Psi(r, x)\}$.

In particular, the relation "$x \in W_r$" is first-order expressible in the language of set theory.

We are able to treat the collection $\{W_r : r \in V\}$ of grounds in a first-order fashion.

- Is there a ground W in which the Continuum Hypothesis (CH) holds?
- Whether every ground satisfies some statement.
- Whether every ground has a proper ground.
- The intersection of all grounds.

These are second-order objects and second-order statements of set-theory.
However, the uniform definability allows us to describe such statements in ZFC.

- One can ask (in ZFC) if $\forall r \in V$, W_r satisfies some statement.
- One can ask (in ZFC) whether $\forall r \exists s (W_s \subsetneq W_r)$.
- One can define (in ZFC) the class $\{x : \forall r (x \in W_r)\}$.

Some geological notions

Definition

1. We say that the grounds W_1, W_2 are downward directed if there is $r \in V$ such that $W_r \subseteq W_1 \cap W_2$. The downward directed grounds hypothesis (DDG), is the sentence
$$\forall r \forall s \exists t (W_t \subseteq W_r \ \wedge \ W_t \subseteq W_s).$$
2. The ground models of the universe are downward set-directed if for every set $I \in V$, there is $r \in V$ such that $W_r \subseteq \bigcap_{i \in I} W_i$. The strong downward-directed grounds hypothesis (sDDG) is the sentence
$$\forall x \exists s \forall r \in x (W_s \subseteq W_r).$$
3. We say that there are only set many grounds if there is a set $I \in V$ such that $\forall s \exists r \in I (W_r = W_s)$. If there is no such I, then there are proper class many grounds.

Some geological notions

Definition

A ground model W_r is a bedrock model if $\forall s(W_s \subseteq W_r \to W_s = W_r)$.
The Bedrock Axiom (BA) is the assertion

$$\exists r \forall s(W_s \subseteq W_r \to W_s = W_r).$$

Question

If the Bedrock Axiom holds, is the bedrock model unique?

Reducing Second to First order

The Ground Axiom is equivalent to the first-order assertion

$$\forall r \forall x (x \in W_r).$$

Can we force GA? Yes.
We need to develop the more powerful (but less tractable) **class forcing** technique.

Class Forcing

- Why is it less tractable? Class forcing can destroy ZFC axioms.
- Why is it more powerful?
    - We can modify the GCH pattern unboundedly.
    - We can prove the consistency of the negation of the Bedrock Axiom (BA).

A failure of Replacement

Counterexample 1: collapse forcing

$\mathrm{Col}(\omega, ORD)$ denotes the partial order whose conditions are finite partial functions from a subset of ω to ORD, ordered by reverse inclusion. That is,

- $p \in \mathrm{Col}(\omega, ORD)$ iff $p : a \to ORD$, where $a \subset \omega$ is finite, and
- $\forall p, q \in \mathrm{Col}(\omega, ORD)\big(p \leq q \leftrightarrow p \supseteq q\big)$.

The forcing $\mathrm{Col}(\omega, ORD)$ adds a surjective function $\omega \to ORD$. In particular, **Replacement** fails in the generic extension.

But it can get even worse.

The class version of the Lévy collapse

Counterexample 2: the class version of the Lévy collapse

For each $\gamma \in ORD \cup \{ORD\}$ we denote by $\mathrm{Col}(\omega, <\gamma)$ the partial order whose conditions are functions $p : \mathrm{dom}(p) \to \gamma$ satisfying

- $\mathrm{dom}(p)$ is a finite subset of $\gamma \times \omega$,
- for all $\langle \alpha, n \rangle \in \mathrm{dom}(p)$, $p(\alpha, n) < \alpha$,

ordered by reverse inclusion.
The forcing $\mathrm{Col}(\omega, <ORD)$ adds a surjective function $f_\gamma : \omega \to \gamma$, for each $\gamma \in ORD$. Then there is no $\mathrm{Col}(\omega, <ORD)$-name σ for the power set of ω. Hence **Power set axiom** fails in the generic extension.

ZFC preservation

- How can we prevent this pathology? We need to place some restrictions on the class-sized partial orders used for forcing.
- The key point is that $\mathrm{Col}(\omega, ORD)$ and $\mathrm{Col}(\omega, < ORD)$ are not tame.

Pretamenes

Definition (Pretameness)

A p.o. P is pretame if and only if whenever $\langle D_i : i \in a \rangle$ is a sequence of dense classes, $a \in M$ and $p \in P$, then there exist a condition $q \leq p$ and a sequence $\langle d_i : i \in a \rangle \in M$ such that $d_i \subset D_i$ and d_i is predense $\leq q$ for each $i \in a$.

Pretameness allows us to pass from dense classes to predense sets by strengthening a given condition.

Pretameness is equivalent to **ZFC-Power set preservation**.

To handle the Power set preservation we need **tameness**.

Tameness

A predense $\leq p$ partition is a pair (D_0, D_1) such that $D_0 \cup D_1 \subset P$ is predense $\leq p$ and $(p_0 \in D_0 \ \wedge \ p_1 \in D_1) \rightarrow (p_0, p_1$ are incompatible). Let $\langle (D_0^i, D_1^i) : i \in a \rangle$ and $\langle (E_0^i, E_1^i) : i \in a \rangle$ be sequences of predense $\leq p$ partitions, where $a \in M$. We say that they are equivalent $\leq p$ if for each $i \in a$, $\{q \in P : q \text{ meets } D_0^i \leftrightarrow q \text{ meets } E_0^i\}$ is dense $\leq p$.

Definition (Tameness)

P is tame if and only if P is pretame and for each $a \in M$ and $p \in P$ there are $q \leq p$ and $\alpha \in ORD^M$ such that whenever $\mathcal{D} = \langle (D_0^i, D_1^i) : i \in a \rangle \in M$ is a sequence of predense $\leq q$ partitions, $\{r \in P : \mathcal{D}$ is equivalent $\leq r$ to some $\mathcal{E} = \langle (E_0^i, E_1^i) : i \in a \rangle \in M_\alpha\}$ is dense $\leq q$.

It turn out that

tameness is equivalent to **ZFC preservation**.

From now on we work with forcing notions which are tame. Thus, ZFC in the generic extension is safe.

A tame p.o. forcing GA

Theorem

If M satisfies ZFC, then there is a forcing extension of M by **class forcing** which satisfies ZFC+GA.

We need some preliminary results before proving the Theorem:

- the Continuum Coding Axiom.
- Easton's Theorem.

The Continuum Coding Axiom

We make use of a new axiom.

Definition

The Continuum Coding Axiom (CCA) is the assertion that for every ordinal α and for every set $a \subset \alpha$ there is an ordinal θ such that

$$\beta \in a \leftrightarrow 2^{\aleph_{\theta+\beta+1}} = (\aleph_{\theta+\beta+1})^+ \text{ for every } \beta < \alpha.$$

Theorem (CCA $\to$ GA)

The Continuum Coding Axiom implies the Ground Axiom.

Proof

Suppose $V \models$ CCA and $V = W[G]$ for some $G \subset P \in W$, which is P-generic over W. Since the GCH pattern is not affected above $|P|$, it follows that every set in V is coded into W, and so $V \subset W$. Hence W is a trivial ground.

Making sets definable

The second ingredient is Easton's Theorem.

Example

Suppose $x \subset \omega$ is a set of natural numbers. Perhaps $x \in M$ is not definable in M. Can we make it definable in a forcing extension?

Yes. Easton's Theorem gives us complete control over the GCH pattern on a set regular cardinals. So we may find a forcing extension $M[G]$ in which

$$n \in x \text{ if and only if } 2^{\aleph_n} = \aleph_{n+1}.$$

Thus, the set x becomes definable in $M[G]$.

We want to iterate this idea to make every set definable from ordinal parameters.

Forcing CCA

Theorem (Con(CCA))

If $M \models$ ZFC, then there is a forcing extension of M by class forcing which satisfies ZFC+CCA.

Nontrivial grounds

- Now suppose ¬GA is true in V. If so, it makes sense to explore the 'geology' of V and we could search for the possible grounds $W \subset V$.
- Is there a ground W such that $W \models$ GA? Equivalently, does BA hold in V?
- Forcing extensions of a model of GA are models of BA. Are there any other models? Yes.

Theorem (Con(¬BA))

If $M \models$ ZFC, then there is a forcing extension of M by class forcing which satisfies ZFC+¬BA.

Consistency of $\neg$BA, a sketched proof

We may assume $M \vDash$ CCA. We define in M a class forcing notion P.

- Let $R = \{\lambda : \lambda \text{ is regular} \wedge 2^{<\lambda} = \lambda\}$.
- For each $\lambda \in R$ let $Q_\lambda = \mathrm{Add}(\lambda, 1)$.
- P is the product $\prod_{\lambda \in R} Q_\lambda$ with Easton support on R.

We claim that $M[G] \vDash \neg$BA, where G is P-generic over M. Let W be a ground of $M[G]$.

- By CCA, we get that $M \subset W$.
- For $\delta \in R$ big enough, $M[G^{>\delta}] \subset W$, where $G^{>\delta}$ is $(\prod_{\lambda > \delta} Q_\lambda)$-generic over M.

Therefore,

$$M[G^{>\delta}] \subset W \subset M[G].$$

This shows that no ground W of $M[G]$ satisfies GA and so $M[G] \vDash \neg$BA.

Some consequences of the proof

Remark 1

The extension $M[G]$ has class many grounds.

Is it possible to have $\neg$BA with only set many grounds?

Remark 2

The proof shows that there are no grounds below M, i.e. $M \subset W \subset M[G]$, whenever W is ground of $M[G]$.
In particular, M is contained in the intersection of all grounds of $M[G]$.

Can we manipulate a given model in such a way that also the converse inclusion is true?

The mantle: removing an entire strata of forcing

Moving inwards we meet the mantle.

Definition

The mantle of V, denoted $\mathfrak{M}^V$ is the intersection of all grounds of V.

By the uniform definability of grounds, $\mathfrak{M}^V$ is a first-order **definable** transitive class. Indeed,

$$x \in \mathfrak{M}^V \text{ if and only if } \forall r(x \in W_r).$$

- What more can we say about $\mathfrak{M}^V$?
- We might expect it to have some nice structural or combinatorial properties.
- Although this position seems to be highly appealing, our main Theorem provides strong evidence against it.

Manipulating the mantle

Theorem

Every model M of ZFC is the mantle of another model of ZFC.

Conclusion

- Anything that can occur in a model of ZFC, can also occur in the mantle.
- So, by sweeping away the accumulated sands of forcing, what we find is not a highly regular structure, but rather an arbitrary model of set theory.
- It follows that we cannot expect to prove any regularity features about the mantle.

A sketch of the proof

For each $\alpha \in ORD$, let $\delta_\alpha = \beth^+_{\omega\cdot(\alpha+1)}$ and define $Q_\alpha \in M$ to be the lottery sum of

- a forcing that forces $2^{\delta_\alpha} = \delta_\alpha^+$, namely $\mathrm{Add}(\delta_\alpha^+, 1)$,
- a forcing that forces $2^{\delta_\alpha} \neq \delta_\alpha^+$, namely $\mathrm{Add}(\delta_\alpha, (2^{<\delta_\alpha})^{++})$.

That is,

$$Q_\alpha = \mathrm{Add}(\delta_\alpha^+, 1) \bigoplus \mathrm{Add}(\delta_\alpha, (2^{<\delta_\alpha})^{++}).$$

We force with the product $P = \prod_{\alpha \in ORD} Q_\alpha$ with set support. If G is P-generic over M, we have to show that

$$\mathfrak{M}^{M[G]} = M.$$

The mantle of the target model

The mantle can be far from, or close to the universe.

- M has a **class** forcing extension $M[G]$ such that $M[G]$ is not a **set** forcing extension of its mantle.
- M has a class forcing extension $M[G]$ such that $M[G]$ has no proper ground, so $\mathfrak{M}^{M[G]} = M[G]$.

Geologies

"[. . .] we should like briefly to mention and then leave for the future the idea of undertaking the entire set-theoretic geology project of this article in the more general context of pseudo-grounds, rather than only the set-forcing grounds, for this context would include these other natural extensions that are not a part of the current investigation."[1]

[1] GUNTER FUCHS, JOEL DAVID HAMKINS, AND JONAS REITZ, *SET-THEORETIC GEOLOGY*.

Cover and approximation properties

In ZFC, Hamkins' cover and approximation properties are important tools for proving the uniform definability of grounds.

Definition

Suppose that W, V are transitive models of ZFC, δ is a cardinal in V and $W \subseteq V$.

1. W satisfies the δ-cover property for V if for each $A \in V$ with $A \subset W$ and $|A|^V < \delta$ there is a set $B \in W$ with $A \subset B$ and $|B|^W < \delta$.
2. W satisfies the δ-approximation property for V if for each $A \in V$ with $A \subset W$, if $A \cap B \in W$ for every $B \in W$ with $|B|^W < \delta$, then $A \in W$.

Fact (Hamkins)

Every ground satisfies the δ-cover and δ-approximation properties for some δ.

Each ground W can be defined as follows.
For a big enough cardinal δ, let $X = \mathcal{P}(\delta^+)^W$. Then W is definable with the parameter $r = \langle \delta, X, P, G \rangle$ as the **unique** model M satisfying

- the δ-cover and δ-approximation properties for V,
- $\mathcal{P}(\delta^+)^M = X$,
- $V = M[G]$.

Pseudo-grounds

We consider a generalization of the idea of geology to other extensions.

Definition (pseudo-ground)

Suppose $U \subset V$ are transitive models of ZFC. U is a pseudo-ground of V if there is a regular cardinal δ such that

1. $(\delta^+)^U = (\delta^+)^V$,
2. U satisfies the δ-cover property for V,
3. U satisfies the δ-approximation property for V.

Remark

If W is a ground of V and $P \in W$ is the witnessing forcing notion, then W satisfies the δ-approximation and δ-cover properties for V for any regular δ with $\delta > |P|$. Hence, grounds are pseudo-grounds.

The pseudo-ground-model definability theorem

We still have first-order definable access to the family of pseudo-grounds of the universe.

Theorem

There exists a formula $\Theta(y, x)$ such that

1. For each $s \in V$, the class $U_s = \{x : \Theta(s, x)\}$ is a pseudo-ground of V and $s \in U_s$.
2. For every pseudo-ground U of V, there is a parameter $s \in U$ such that $U = \{x : \Theta(s, x)\}$.

In particular, the relation "$x \in U_s$" is first-order definable in the language of set theory.

A new geology

Because the definability Theorem applies in this more general case, we may formalize the pseudo-ground analogues of the Ground Axiom, the Bedrock Axiom and the mantle.

Definition

- The pseudo-Ground Axiom (pGA) asserts that the universe has no nontrivial pseudo-grounds. That is,
$$\forall s(U_s = V).$$
- U_s is a pseudo-bedrock if $\forall t(U_t \subset U_s \to U_t = U_s)$. The pseudo-Bedrock Axiom (pBA) asserts that there exists a pseudo-bedrock.
- The pseudo-mantle $p\mathfrak{M}^V$ of V is the intersection of all of its pseudo-grounds. Equivalently,
$$p\mathfrak{M}^V = \{x : \forall s(x \in U_s)\}.$$

Class forcing extensions

Our aim is to explore some connections between:

- Pseudo-grounds.
- Class forcing extensions.

So far we know that a model $M \models \mathrm{ZFC}$ is definable in its **set** forcing extension $M[G]$.

Question

Does the same hold for **class** forcing?

In general, the answer is negative (Antos).

Towards a positive answer

Definition

Let δ be a regular cardinal. A forcing notion P has a closure point at δ when it factors as $P \cong Q * \dot{R}$ where Q is a nontrivial poset, $|Q| \leq \delta$ and $1_Q \Vdash_Q$ "$\dot{R}$ is δ^+-closed".

Theorem (The class ground-model definability Theorem)

Let $P \subset M$ be a forcing notion and let G be P-generic over M. If there exists a regular cardinal δ in M such that

1. P has a closure point at δ, and
2. P does not collapse δ^{++},

then M is a pseudo-ground of $M[G]$.

Applications: forcing notions with a closure point

We may use the Theorem to manipulate the structure of the pseudo-mantle by class forcing.

- Let Q_α denotes in M the lottery sum $\mathrm{Add}(\delta_\alpha^+, 1) \bigoplus \mathrm{Add}(\delta_\alpha, (2^{<\delta_\alpha})^{++})$ for each $\alpha \in ORD$, where $\delta_\alpha = \beth^+_{\omega\cdot(\alpha+1)}$.
- Then the product $P = \prod_{\alpha \in ORD} Q_\alpha$ can be factored as a poset of size $< \delta_\alpha$ followed by a δ_α-closed tail forcing.
- We know that $M = \mathfrak{M}^{M[G]}$ for some P-generic G over M. Moreover, by the Theorem, M is a pseudo-ground of $M[G]$.
- Consequently, $p\mathfrak{M}^{M[G]} \subset M = \mathfrak{M}^{M[G]}$.

A similar argument shows the consistency of GA+¬pGA.

Theorem

Let M be a model of ZFC. Then there exists a class forcing extension $M[G]$ satisfying GA+¬pGA. In particular, $p\mathfrak{M}^{M[G]} \subsetneq \mathfrak{M}^{M[G]}$.

How free are we to manipulate the pseudo-mantle?

Besides the natural definition of the pseudo-mantle, there are many open questions. An important one is whether we may control the pseudo-mantle of the target model.

Question 1

Let M be a model of ZFC.

- Is there a class forcing extension $M[G]$ such that $p\mathfrak{M}^{M[G]} = M[G]$ or, equivalently, $M[G] \models \text{pGA}$?
- Is there a class forcing extension $M[H]$ such that $p\mathfrak{M}^{M[H]} = M$?

Our most fundamental lack of knowledge about the pseudo-mantle is that we do not yet know whether or not it is necessarily a model of ZFC.

Question 2

Does the pseudo-mantle satisfy ZF? Or ZFC?

Thank you for your attention!